

Sheaves and Affine Varieties

Mumford, *Red Book*, Ch. I, §4

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Definitions

Definition 1

Let X be a topological space. A presheaf (of sets) $\mathcal{F}$ on X consists of the following data:

- for all open sets $U \subset X$, a set $\mathcal{F}(U)$
- for all pairs of open sets $U \subset V$, a map $\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$

such that

- ① $\rho_U^U = \text{id}_{\mathcal{F}(U)}$ for all open U
- ② If $U \subset V \subset W$, then $\rho_U^W = \rho_U^V \circ \rho_V^W$.

Remark 2

We call the maps ρ_U^V restriction maps. We call the sets $\mathcal{F}(U)$ the sections of $\mathcal{F}$ over U . We often write $\rho_U^V(s)$ as just $s|_U$ when it is clear where the section s lives.

Definitions

Definition 3

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X . A morphism of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a collection of maps $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each open U such that if $U \subset V$, we have

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ \downarrow \rho_U^V & & \downarrow \rho_U^V \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array}$$

commutes.

Definition 4

A presheaf $\mathcal{F}$ on X is a *sheaf* if for every open set U and open cover $U = \bigcup_i U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\quad r \quad} \prod_i \mathcal{F}(U_i) \begin{array}{c} \xrightarrow{\quad p \quad} \\ \xrightarrow{\quad q \quad} \end{array} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer. Concretely, the restriction map

$$r: \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_i,$$

is injective with image exactly those families (s_i) satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j , i.e. those on which the two maps p, q agree.

Definitions

Remark 5

The maps p and q are the same thing morally: you take some $(s_i)_i$ and for every (i, j) you either restrict s_i to $U_i \cap U_j$ or s_j to $U_i \cap U_j$. These are really the only natural things to do in this case. The sheaf condition says that all of the (s_i) that give you the same thing no matter what order you do it must come from $\mathcal{F}(U)$.

Remark 6

We can concretely rewrite the sheaf condition. The first map being injective just means that for all $s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ for all i , then $s = t$. The second requirement just means that if we have some collection of $s_i \in \mathcal{F}(U_i)$ such that for all i, j , $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then we have some $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Question 7

What can we say about this s that must exist in the second condition?

Examples

Example 8 (a sheaf)

Fix topological spaces X, Y . For open $U \subseteq X$, let $\mathcal{F}(U)$ be the continuous maps $U \rightarrow Y$, with restriction of domain. This is a sheaf: continuous maps agreeing on overlaps define one map on the union, and continuity can be checked on the open cover (the pasting lemma).

Example 9 (a presheaf that is not a sheaf)

Let $X = \mathbb{R}$ and let $\mathcal{F}(U)$ be the *bounded* continuous functions $U \rightarrow \mathbb{R}$. Restriction preserves boundedness, so $\mathcal{F}$ is a presheaf. But cover $\mathbb{R}$ by $U_n = (-n, n)$ and take $f_n = \text{id}|_{U_n}$: each f_n is bounded and they agree on overlaps, yet the only function gluing them is $x \mapsto x$, which is unbounded. Gluing fails, so $\mathcal{F}$ is not a sheaf.

Remark 10

Note $\mathcal{F}$ is still separated (functions agreeing on a cover are equal); it is existence of the glued section, not uniqueness, that fails.

Definitions

Definition 11

Let $\mathcal{F}$ be a (pre)sheaf on X . Let $x \in X$. The collection of $\mathcal{F}(U)$ for $U \ni x$ open forms a directed system. We can form

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$$

called the stalk of $\mathcal{F}$ at x .

Remark 12

Notice in the case of sets that $\mathcal{F}_x$ is just pairs (U, s) where $U \ni x$ is open and $s \in \mathcal{F}(U)$ under the equivalence that $(U, s) \sim (V, t)$ if and only if there is some open $x \in W \subset U \cap V$ such that $s|_W = t|_W$.

Question 13

Let $\mathcal{F}$ be the sheaf of holomorphic functions on $\mathbb{C}$. What is the stalk at a point?

Sheafification

Definition 14

A *sheafification* of a presheaf $\mathcal{F}$ is a sheaf $\mathcal{F}^+$ with a universal morphism $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$: every morphism from $\mathcal{F}$ into a sheaf $\mathcal{G}$ factors uniquely through θ .

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad \forall \quad} & \mathcal{G} \\ \searrow \theta & & \nearrow \exists! \\ & \mathcal{F}^+ & \end{array}$$

Theorem 15

$\mathcal{F}^+$ exists, is unique up to unique isomorphism, has the same stalks $\mathcal{F}_x \cong \mathcal{F}_x^+$, and θ is an isomorphism if and only if $\mathcal{F}$ is a sheaf.

Aside 16

Concretely, $\mathcal{F}^+(U) =$ compatible germs $s: U \rightarrow \coprod_x \mathcal{F}_x$ locally from $\mathcal{F}$.

Examples

Example 17 (direct image)

Let $f: X \rightarrow Y$ be continuous and $\mathcal{F}$ a sheaf on X . Since $f^{-1}(U)$ is open for open $U \subseteq Y$, the rule $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ defines a sheaf on Y , the direct image $f_*\mathcal{F}$ as it inherits the axioms from $\mathcal{F}$.

Example 18 (sheafifying the bounded functions)

Take the presheaf $\mathcal{F}$ of bounded continuous functions on $\mathbb{R}$ from before. A function is locally bounded-and-continuous exactly when it is continuous (continuous functions are locally bounded), so the local-section description gives

$$\mathcal{F}^+(U) = \{ \text{continuous } U \rightarrow \mathbb{R} \},$$

the sheaf of all continuous functions. Sheafification just drops the non-local “bounded” condition; θ is the inclusion, isomorphic on every stalk, but not globally (e.g. $\text{id}_{\mathbb{R}}$).

Definition 19

Let $X \subset k^n$ be an irreducible algebraic set and R its coordinate ring. Since X is irreducible, $I(X)$ is a prime ideal and R is an integral domain. Let K be $\text{Frac}(R)$. For $x \in X$, let $\mathfrak{m}_x = \{f \in R : f(x) = 0\}$. This is a maximal ideal (it is the kernel of $\text{ev}_x: R \rightarrow k, g \mapsto g(x)$, which is onto, so $R/\mathfrak{m}_x \cong k$). Let $\mathcal{O}_{X,x} = R_{\mathfrak{m}_x}$. We see $\mathcal{O}_{X,x} = \{f/g : f, g \in R, g(x) \neq 0\} \subset K$. For $U \subset X$ open, let

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$$

If $V \subset U$, we have $\mathcal{O}_X(U) \subset \mathcal{O}_X(V)$ and if we take the inclusion as the restriction map, this defines a sheaf $\mathcal{O}_X$.

Remark 20

We can view elements of $\mathcal{O}_X(U)$ as functions on U as follows: if $F \in \mathcal{O}_X(U)$ and $x \in U$, then $F \in \mathcal{O}_{X,x}$, so we can write $F = f/g$ where $g(x) \neq 0$. We can define $F(x) = f(x)/g(x)$. If $F(x) = 0$ for all $x \in U$, then $F = 0$ (a nonempty open of an irreducible space is dense and a regular function is determined by its values, so the numerator vanishes on X , i.e. is 0 in R). Hence, we can identify $\mathcal{O}_X(U)$ with the associated ring of functions on U .

Proposition 21

Let X be an irreducible algebraic set and let $R = \Gamma(X)$. Let $f \in R$ and $X_f = \{x \in X : f(x) \neq 0\}$. Then $\mathcal{O}_X(X_f) = R_f$.

Remark 22

Recall $\mathcal{O}_X(U) = \bigcap_{x \in U} R_{\mathfrak{m}_x} \subseteq K := \text{Frac}(R)$, and X_f is the distinguished open where f does not vanish. The inclusion $\supseteq$ is easy; the content is that no “extra” functions appear on X_f .

Proof of the Proposition.

($\supseteq$) For $x \in X_f$ we have $f(x) \neq 0$, so $f \notin \mathfrak{m}_x$; thus f is a unit in $R_{\mathfrak{m}_x}$ and $R_f \subseteq R_{\mathfrak{m}_x}$. Intersecting over all $x \in X_f$ gives $R_f \subseteq \mathcal{O}_X(X_f)$.

($\subseteq$) Let $h \in \mathcal{O}_X(X_f)$ and set $\mathfrak{d} = \{a \in R : ah \in R\}$, an ideal of R . For $x \in X_f$, the membership $h \in R_{\mathfrak{m}_x}$ lets us write $h = a/s$ with $s \notin \mathfrak{m}_x$; then $s \in \mathfrak{d}$ but $s \notin \mathfrak{m}_x$, so $\mathfrak{d} \not\subseteq \mathfrak{m}_x$, i.e. $x \notin V(\mathfrak{d})$. Hence $V(\mathfrak{d}) \cap X_f = \emptyset$, so $V(\mathfrak{d}) \subseteq V(f)$. By the Nullstellensatz $f \in \sqrt{\mathfrak{d}}$, so $f^n \in \mathfrak{d}$ for some $n \geq 1$; then $f^n h \in R$ and

$$h = \frac{f^n h}{f^n} \in R_f.$$



Corollary 23

$$\Gamma(X, \mathcal{O}_X) = R$$

Corollary 24

If $f \in \mathcal{O}_X(U)$ and f vanishes nowhere on U , then $1/f \in \mathcal{O}_X(U)$.

Corollary 25

The stalk of $\mathcal{O}_X$ at x is $\mathcal{O}_{X,x}$.

Algebraic Sets

Remark 26

The fraction field $K = \text{Frac}(R)$ is the *field of rational functions* $k(X)$. Every $\mathcal{O}_X(U)$ (for $U \neq \emptyset$) and every local ring $\mathcal{O}_{X,x} = R_{\mathfrak{m}_x}$ sits inside it, and $K = \bigcup_{U \neq \emptyset} \mathcal{O}_X(U)$: a rational function is one that is regular on *some* nonempty open. So the field is recovered intrinsically from the sheaf.

Warning 27

An element of $\mathcal{O}_X(U)$ is only locally a single quotient f/g : at each $x \in U$ we can write it as f/g with $g(x) \neq 0$, but (when R is not a UFD) one pair (f, g) need not work on all of U . On a distinguished open X_f it does, by the Proposition/ (Mumford has a footnote explaining an example where no global f/g exists)

Algebraic Sets

Proposition 28

Let $X \subset k^n$ and $Y \subset k^m$ be irreducible algebraic sets, and let $f : X \rightarrow Y$ be a continuous map. TFAE:

- ① f is a morphism
- ② for all $g \in \Gamma(Y, \mathcal{O}_Y)$, $g \circ f \in \Gamma(X, \mathcal{O}_X)$
- ③ for all open $U \subset Y$, and $g \in \Gamma(U, \mathcal{O}_Y)$, we have $g \circ f \in \Gamma(f^{-1}(U), \mathcal{O}_X)$
- ④ for all $x \in X$ and $g \in \mathcal{O}_{Y, f(x)}$, we have $g \circ f \in \mathcal{O}_{X, x}$

Proof idea.

(i)–(iii) just unwind “pullback of a section is a section.” The only real content is passing from the global/affine statement to the stalk-local one (iv), which is $\mathcal{O}_X(X_f) = R_f$ again. $\square$

Affine Varieties

Definition 29

An affine variety is a topological space X plus a sheaf of k -valued functions $\mathcal{O}_X$ on X which is isomorphic to an irreducible algebraic subset of some k^n plus the sheaf defined before.

Definition 30

The affine variety $(k^n, \mathcal{O}_{k^n})$ is $\mathbb{A}^n$, affine n -space.

Bijection $\neq$ isomorphism

In general, a bijection $f : X \rightarrow Y$ of varieties may correspond to an isomorphism of coordinate ring of Y with a proper subring of X .

Example 31

Let $\text{char}(k) = p \neq 0$. Define a morphism $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ by $f(t) = t^p$. This is a bijection (since k is algebraically closed and thus perfect). On the ring level, this is an inclusion $k[X^p] \rightarrow k[X]$ which is NOT an isomorphism.

Example 32

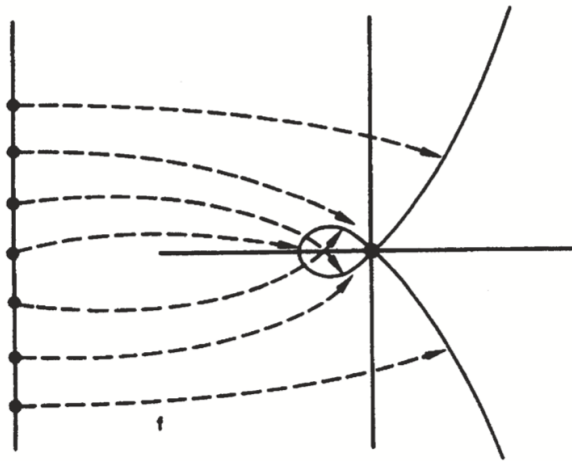
Let k be algebraically closed. Define a morphism $f : \mathbb{A}^1 \rightarrow \mathbb{A}^2$ by $f(t) = (t^2, t^3)$. The image of this morphism is the irreducible closed curve $C : x^3 = y^2$. This map is a bijection from $\mathbb{A}^1 \rightarrow C$, yet only corresponds to the inclusion of rings $k[T^2, T^3] \rightarrow k[T]$.

Bijjective $\neq$ isomorphism

Example 33

Let $f : \mathbb{A}^1 \rightarrow \mathbb{A}^2$ be given by $X = t^2 - 1$ and $Y = t(t^2 - 1)$. The image of this morphism is the curve $D : Y^2 = X^2(X + 1)$. Notice that f is a bijection between $\mathbb{A}^1$ and D except at the points $t = -1$ and $t = 1$ which both map to the origin. Let $X_1 = \mathbb{A}^1 - 1$ which is an affine variety with coordinate ring $k[T, (T - 1)^{-1}]$. Then f restricts to a bijection $f' : X_1 \rightarrow D$. This gives us an inclusion of rings $k[T^2 - 1, T(T^2 - 1)] \rightarrow k[T, (T - 1)^{-1}]$ which cannot be an isomorphism. Visually,

Bijjective $\neq$ isomorphism



Open and closed subvarieties

Definition 34

Let Y be an irreducible closed subset of an affine variety $(X, \mathcal{O}_X)$. We define an induced sheaf $\mathcal{O}_Y$ on Y as follows: for $V \subseteq Y$ open,

$$\mathcal{O}_Y(V) = \left\{ f: V \rightarrow k \mid \begin{array}{l} \text{for each } x \in V \text{ there is an open } U \ni x \text{ in } X \\ \text{and } F \in \mathcal{O}_X(U) \text{ with } f = F|_{U \cap V} \end{array} \right\}.$$

Proposition 35

$(Y, \mathcal{O}_Y)$ is an affine variety.

Proof sketch.

Y is itself an irreducible algebraic subset of k^n with coordinate ring $R/I(Y)$. The induced $\mathcal{O}_Y$ agrees with the structure sheaf of Y as an algebraic set (sections of $\mathcal{O}_X$ restrict to Y , and locally every regular function on Y so arises), so $(Y, \mathcal{O}_Y)$ is affine. □

Open and closed subvarieties

Proposition 36

Let $(X, \mathcal{O}_X)$ be an affine variety, and let $f \in \Gamma(X, \mathcal{O}_X)$. Then $(X_f, \mathcal{O}_X|_{X_f})$ is an affine variety.

Proof sketch.

Embed X_f as the graph $\Gamma = \{(x, t) \in X \times \mathbb{A}^1 : tf(x) = 1\} \subseteq k^{n+1}$, an irreducible algebraic set with coordinate ring $R[t]/(tf - 1) \cong R_f$. The projection $(x, t) \mapsto x$ is a bijection $\Gamma \rightarrow X_f$ with inverse $x \mapsto (x, 1/f(x))$, and it matches the structure sheaves (both have ring R_f , using $\mathcal{O}_X(X_f) = R_f$). So $(X_f, \mathcal{O}_X|_{X_f}) \cong \Gamma$ is affine. □

Open and closed subvarieties

Remark 37

Not all open subsets of affine varieties are affine varieties themselves. For example, $\mathbb{A}^2 - (0, 0)$ is NOT affine. Cover $U = \mathbb{A}^2 \setminus \{0\}$ by $D(x)$ and $D(y)$; a global section restricts into $k[x, y]_x$ and $k[x, y]_y$, so

$$\Gamma(U, \mathcal{O}) = k[x, y]_x \cap k[x, y]_y = k[x, y] = \Gamma(\mathbb{A}^2, \mathcal{O})$$

inside $k(x, y)$. If U were affine it would be determined by its ring $k[x, y]$, hence isomorphic to $\mathbb{A}^2$ with the inclusion $U \hookrightarrow \mathbb{A}^2$ inducing the identity on $k[x, y]$ forcing $U = \mathbb{A}^2$, contradicting that the origin is missing.