Advanced Topics Course: Algebraic Geometry

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Contents

Note that this class was taught Alena Pirutka in Spring 2024 at NYU. I have added my own details and remarks throughout to help the less than intelligent student (i.e. me). Any errors in these notes are 100% my fault.

1 Lecture on 3/26

The plan over all will be to cover

1. Schemes
2. Properties of schemes and morphisms
3. Cohomology

The plan for this lecture will be to cover

1. A few reminders on affine schemes
2. Schemes in general and the Proj construction
3. Morphisms, points, open and closed immersions.
4. (If time permits) Morphisms of finite type

1.1 Reminders

We will always use commutative unital rings in this course (We will just call them rings). Our main interests will be in studying rings of the form $\frac{k[x_1,\ldots,x_n]}{I}$, where k is a field, I is an ideal of some ring A.

Definition 1.1. *Let A be a ring. Then* $\mathrm{Spec}(A) := \{\mathfrak{p} \in A : \mathfrak{p}$ *is a prime ideal*$\}$

We can define a topology on $\mathrm{Spec}(A)$, called the Zariski topology, where we define the closed sets to be of the form $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subset \mathfrak{p}\}$, where I is just an ideal of A. Moreover, sets of the form $D(f) := \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$, where f is any element of our ring A. We call these basic or distinguished open sets. They form a basis for our topological space on $\mathrm{Spec}(A)$. Next, we define a sheaf of rings on this topological space $\mathrm{Spec}(A)$. We recall in general the definition of sheaves of rings (or sets, abelian groups, etc).

Definition 1.2. *Let X be a topological space. A sheaf of rings on X is the following data*

- *For each open subset $U \subset X$, a ring $\mathbf{F}(U)$*
- *For each inclusion of open subsets $U \subset V$, a ring homomorphism called a restriction map:*

$$\begin{aligned} r_{VU} : \mathbf{F}(V) &\to \mathbf{F}(U) \\ s &\mapsto s|_U \end{aligned}$$

Such that the following hold:

1. $\mathbf{F}(\emptyset) = 0$
2. r_{UU} *is the identity map* $\mathbf{F}(u) \to \mathbf{F}(U)$
3. *For all inclusions of open sets* $W \subset V \subset U$*, we have that* $r_{UW} = r_{VW} \circ r_{UV}$

This defines a presheaf of rings on X*. A sheaf of rings on* X *satisfy the two additional conditions:*

- *If* U *is an open set,* $\{U_i\}$ *is an open cover of* U*, and* $s, t \in \mathbf{F}(U)$ *are elements such that* $s|_{U_i} = t|_{U_i}$ *for all* i*, then* $s = t$
- *If* U *is an open set,* $\{U_i\}$ *is an open cover of* U*, and we have elements* $s_i \in \mathbf{F}(U_i)$ *such that for all* i, j*,* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$*, then there exists* $s \in \mathbf{F}(U)$ *such that* $s|_{U_i} = s_i$ *for all* i*.*

Definition 1.3. *Given a sheaf of rings* $\mathcal{O}_X$ *on a space* X*. We define the stalk of* $\mathcal{O}_X$ *at* $x \in X$ *to be*

$$\mathcal{O}_{X,x} := \varinjlim \mathcal{O}_X(U)$$

where we index over all open sets U *which contain* x*.*

Definition 1.4. *Let* X *be a topological space with* $\mathcal{O}_1, \mathcal{O}_2$ *presheaves on* X*. A morphism of presheaves is a ring homomoprhism* $\phi(U) : \mathcal{O}_1(U) \to \mathcal{O}_(U)$ *of each open subset* $U \subset X$ *such that for an open inclusion* $V \subset U$ *the following diagram commutes*

$$\begin{tikzcd}
\mathcal{O}_1(U) \arrow[r] \arrow[d, "\rho"] & \mathcal{O}_2(U) \arrow[d, "\rho"] \\
\mathcal{O}_1(V) \arrow[r] & \mathcal{O}_V(V)
\end{tikzcd}$$

A morphism of sheaves is just a morphism of presheaves

Remark 1.5. *Recall an element of a stalk is just a pair* (U, s) *with* $s \in \mathcal{O}(U)$*. Our induced map on stalks is just* $\phi(U, s) = (U, \phi(U)(s)$*, which makes a lot of sense if you think about it (keep the same open set and just look at the section in the other ring).*

Remark 1.6. *Morphisms of sheaves is an isomorphism if and only if the all the induced maps on stalks are isomorphisms.*

Warning 1.7. *THIS IS NOT TRUE FOR PRESHEAVES IN GENERAL.*

Now, we define a sheaf of rings on $X = \operatorname{Spec}(A)$. It is enough to define our rings of regular functions (the $\mathbf{F}(U)$) on an open basis. Thus, we define

$$\mathcal{O}_X(D(f)) := A_f$$

We showed in Algebra II this in fact gives us a sheaf on $\mathrm{Spec}(A)$. Moreover, its stalks satisfy

$$\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$$

for all our points in $\mathrm{Spec}(A)$. We call $(X, \mathcal{O}_X)$ a locally ringed space (since $\mathcal{O}_{X,\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A$). (Note that ringed space is just a topological space with a sheaf of rings on it). Next, we recall the definition of morphisms of locally ringed spaces.

Definition 1.8. *Let $(X, \mathcal{O}_X)$,$(Y, \mathcal{O}_Y)$ be locally ringed spaces. By a morphism of locally ringed spaces*

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

We mean an ordered pair $(f, f^{\#})$ where

- *$f : X \to Y$ is a continuous map*
- *$f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$*

such that the induced map of stalks $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a morphism of local rings (note $y = f(x)$).

Remark 1.9. *I personally had a lot of trouble understanding what this induced map of stalks actually was, so I am including this remark on it. First, given any sheaf $\mathcal{O}$ on a space X, a continuous map $f : X \to Y$, and a point $x \in X$, there is a canonical map $(f_*\mathcal{O})_{f(x)} \to \mathcal{O}_x$. This is because an element of $(f_*\mathcal{O})_{f(x)}$ is represented by a section of $f_*\mathcal{O}$ over some open set $V \subset Y$ containing $f(x)$, but this is just a section of $\mathcal{O}$ over $f^{-1}(V)$ which turn determines an element of $\mathcal{O}_x$ (exercise is to show this is compatible with restriction maps). Hence, this is our induced map $(f_*\mathcal{O})_{f(x)} \to \mathcal{O}_x$. Recall, since $f^{\#}$ is a morphism of sheaves, we have a map*

$$f^{\#}_x : \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)}$$

so using the above the discussion we have a map into $\mathcal{O}_{X,x}$ as desired!

We now give an example of this in the case of $\mathrm{Spec}(A)$.

Example 1.10. *Let A, B be rings. We have a bijection*

$$\mathrm{Hom}(A, B) \leftrightarrow \mathrm{Hom}(\mathrm{Spec}(B), \mathrm{Spec}(A))$$

Hence to specify a morphism of affine schemes, it suffices to give a ring homomorphism in the opposite direction. This is because for a map of rings $\phi : A \to B$, we have the map

$$\phi^{-1} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

since the pre-image of prime ideals are prime ideals. Moreover, we see that this takes

$$A_a \to B_{\phi(a)} \text{ by}$$
$$\frac{x}{a^n} \mapsto \frac{\phi(x)}{(\phi(a))^n}$$

where we are looking at how are basic open sets are being mapped. In reverse, we also have a map of rings induced by a map of affine schemes (the global sections). A map

$$\psi : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

gives us a ring homomorphism

$$\phi = \psi|_{A=\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A))}$$

1.2 Schemes In General

Now, we define schemes in general.

Definition 1.11. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $X = \cup X_i$ where the X_i with with the restriction of $\mathcal{O}_X$ to X_i is an affine scheme $(\operatorname{Spec}(A_i), \mathcal{O}_{\operatorname{Spec}(A_i)})$ for some ring A_i.*

Remark 1.12.
- *We call $\mathcal{O}_X(U)$ the ring of regular functions on U*
- *$(\mathcal{O}_X(U))^*$ is the multiplicative group of the invertible functions*

Remark 1.13.
- *The affine open sets are a basis for the topology on X (in that each one is covered by basic open sets)*
- *Let $f \in \mathcal{O}_X(U)$ for some open subset U in X. We can evaluate f at $x \in U$. Explicitly, take an affine subset of U that contains $x = \mathfrak{p}$. We then can look at the equivalence class of f in $\mathcal{O}_{X,x=A_\mathfrak{p}}$, say f_x. Then, $f(x)$ is the image of f_x in $\kappa(x) = \frac{A_\mathfrak{p}}{\mathfrak{p}A}$*
- *An open subset of an affine scheme is NOT necessarily affine. For example, consider the affine plane minus a point: $X = \mathbb{A}_k^2 = k[x, y]$, $U = X - \{0, 0\} = X - \mathfrak{p}$ where $\mathfrak{p} = (x, y)$.*
- *$\mathbb{A}_k^n = \operatorname{Spec}(k[x_1, \ldots, x_n])$ is called affine n-space.*

Now, we construct an example of a non-affine scheme.

Definition 1.14. *A graded ring B is a ring that decomposes into a direct sum*

$$B = \bigoplus_{d \geq 0} B_d$$

of additive abelian groups groups such that for all non-negative integers m, n, $B_m B_n \subset B_{m+n}$. A nonzero element of B_n is said to be homogeneous of degree n.

Remark 1.15. *By the definition of direct sums, every nonzero element $x \in B$ can be uniquely written as the sum $x = b_0 + b_1 + \ldots + b_n$, where each b_i is zero or homogeneous of degree i. We call the nonzero entries the homogeneous coordinates of x.*

Example 1.16. *Let $B = k[x_0, \ldots, x_n]$. Then B_d is just the homogeneous polynomials of degree d.*

Definition 1.17. *We say an ideal $I \subset B$ is homogeneous if it is generated by homogeneous elements. Equivalently, we say it is homogeneous if every element in I has homogeneous components in I.*

In particular, we see that $I = \bigoplus_{d \geq 0}(I \cap B_d)$ and B/I is graded. Also, $(B/I)_d = B_d/(I \cap B_d)$. Lastly, the radical of a homogeneous ideal is homogeneous.

Definition 1.18. *Let $B_+ = \bigoplus_{d>0} B_d$.*

We are now ready to define Proj of a ring.

Definition 1.19. *Let B be a ring. Then*

$$\mathrm{Proj}(B) := \{\mathfrak{p} \subset B : \ \mathfrak{p} \text{ is a prime, homogeneous ideal and } B_+ \not\subset \mathfrak{p}\}$$

Fact 1. *A homogeneous ideal $I \subset B$ with $B_+ \not\subset I$ is prime if and only if for all $a, b \in B$ homogeneous, $ab \in I$ implies a or b is in I.*

We want to give a scheme structure to $\mathrm{Proj}(B)$ in the same way we did for affine schemes. To this end, we define a topology on $\mathrm{Proj}(B)$. We define the closed subsets to be of the form

$$V_+(I) := \{\mathfrak{p} \in \mathrm{Proj}(B) : I \subset \mathfrak{p}\}$$

where $I \subset B$ is homogeneous. We see that

1. $V_+(B_+) = \emptyset$
2. $V_+(0) = \mathrm{Proj}(B)$
3. $V_+(I) \cup V_+(J) = V_+(IJ) = V_+(I \cap J)$
4. $\bigcap_r V_+(I_r) = V_+(\sum_r I_r)$

As before, we have basic/distinguished open subsets that form a basis:

$$D_+(f) := \{\mathfrak{p} \in \mathrm{Proj}(B) : f \notin \mathfrak{p}\}$$

where f is homogeneous. Next, if $\mathfrak{p} \in \mathrm{Proj}(B), f \in B$ homogeneous, we define

$$B_{\mathfrak{p}} := \{\frac{a}{b} : a, b \in B \ \text{ homogeneous of the same degree}, b \notin \mathfrak{p}\}$$
$$B_f := \{\frac{a}{f^n} : a \in B \ \text{ homogeneous of degree n*deg(f)}\}$$

Lemma 1.20. *1.*

$$D_+(f) \xrightarrow{\sim} \operatorname{Spec}(B_{(f)})$$

where the bijection is defined by

$$\mathfrak{p} \mapsto (\mathfrak{p}B_f) \cap B_(f)$$

2. *if f, g are homogeneous of degree > 0 with $D_+(g) \subset D_+(f)$, then the following diagram commutes*

$$\begin{tikzcd} D_+(g) \arrow[r] \arrow[d] & \operatorname{Spec}(B_{(g)}) \arrow[d] \\ D_+(f) \arrow[r] & \operatorname{Spec}(B_{(f)}) \end{tikzcd}$$

3. *If $X = \operatorname{Proj}(B)$, then $\mathcal{O}_X(D_+(f)) = \operatorname{Spec}(B_{(f)})$ defines a sheaf and $\mathcal{O}_{X,\mathfrak{p}} = B_{\mathfrak{p}}$*

Definition 1.21. *A projective variety over a field k is a scheme of type $\operatorname{Proj}(B)$, where $B = \frac{k[x_0,\ldots,x_n]}{I}$ with I a homogeneous ideal.*

Example 1.22. *Let $B = k[x_0, \ldots, x_n]$ be a graded ring (graded by degree of polynomial). We see*

$$B_{x_0} = \{\frac{p(x_0, \ldots, x_n)}{x_0^m} : \text{ deg } p = m\} \qquad = k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$$

Definition 1.23. *The projective n-space over a field k is*

$$\mathbb{P}^n_k := \operatorname{Proj}(k[x_0, \ldots, x_n])$$

We see that $\mathbb{P}^n_k$ can be covered by the $D_+(x_i)$, which is the affine space $\operatorname{Spec}(k[\frac{x_0}{x_i}, \ldots, \frac{\hat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}])$.

Remark 1.24. *The global section of $\mathbb{P}^k_k$ is just k (use the open cover from above and sheave properties to see that they only agree on constant functions on intersections).*

1.3 Morphisms of Schemes

Now, we look at morphisms of schemes in general.

Definition 1.25. *Let X, Y be schemes. A morphism $f : X \to Y$ is a morphism of locally ringed spaces.*

Remark 1.26.
- *f induces a map*

$$f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

which is a map of local rings. This in turn gives an inclusion

$$\kappa(y) \hookrightarrow \kappa(x)$$

where $\kappa(x) = \mathcal{O}_{X,x}/$maximal ideal is the residue field of x.

- *If X, Y are schemes, $X = \bigcup U_i$ is an open cover, then in order to define a map $f : X \to Y$, it is enough to define $f_i : U_i \to Y$ such that all maps coincide on the intersections of the open cover.*

- *For $x \in X$, we have $i_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is a morphism of schemes. This works as follows: if $x \in \operatorname{Spec}(A)$ for some open affine subset, then letting $x = \mathfrak{p}$, $\mathcal{O}_{X,x} = A_\mathfrak{p}$. Thus, we get a map $A \to A_\mathfrak{p}$ which corresponds to i_x.*

Definition 1.27. *Let S be a scheme. An S-scheme is a scheme X equipped with a structure morphism $X \to S$. Let X, Y be S-schemes. A morphism of S-schemes is a morphism $X \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

We say A-scheme when referring to a scheme over $\operatorname{Spec}(A)$.

Example 1.28.
- *Let k be a field. Consider $X = \operatorname{Spec}(\frac{k[x_1,\dots,x_n])}{I}$. We have a map $k \to \frac{k[x_1,\dots,x_n]}{I}$ which includes constants, so X is a k-scheme.*

- *Let X be a scheme over a field k. Let L/k be a field extension. We have an an inclusion map $k \to L$, so we have a map of k-schemes $\operatorname{Spec}(L) \to X$. Then $x \in X$ has image $\mathcal{O}_{X,x} \to L$ which sends $m_x \to 0$. Hence, $\kappa(x) \to L$ is a field extension.*

1.4 Open and Closed Immersions

Definition 1.29. *Let X be a scheme. An open sub-scheme of X is the following:*

- *An open subset $U \subset X$ with*

- *The restriction of the sheaf $\mathcal{O}_X$ to U*

An open immersion is a morphism of schemes $X \to Y$ inducing an isomorphism of X onto an open sub-scheme of Y (ie, a homeomorphism of X onto an open subset of Y along with an isomorphism of sheaves).

Example 1.30. $\operatorname{Spec}(A_f) \to \operatorname{Spec}(A)$

Closed subsets and immersions are not quite as easy to define.

Definition 1.31. *A closed immersion is a morphism of schemes $f : X \to Y$ such that*

- *f induces a homeomorphism from X onto a closed subset in Y*

- *$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective*

A closed sub-scheme of Y is a scheme X together with a closed immersion $i : X \to Y$.

Remark 1.32. *We identify $(X, i), (X', i')$ both closed subsets of Y if there exists an isomorphism of schemes $g : X \to X'$ such that $i' \circ g = i$.*

Theorem 1.33. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and $i : Z \to X$ be a closed immersion. Then Z is affine. In fact, it is isomorphic to the subscheme $\mathrm{Spec}(A/J)$ for some ideal $J \subset A$.*

1.5 Morphisms of Finite Type

We want to study which properties of morphisms can be "checked locally".

Definition 1.34. *We say that a morphism $f : X \to Y$ is local over the base if the following conditions are equivalent*

- *f verifies some property (p)*
- *Y has an open cover $Y = \bigcup V_i$ such that all V_i are affine, $f_i : f^{-1}(V_i) \to V_i$ verifies (p) for all i.*

We recall that a quasi-compact topological space is one in which every open cover has a finite sub-cover (note we do NOT require it do be Hausdorff). Also, we recall that $X = \mathrm{Spec}(A)$ is quasi-compact.

Remark 1.35. *To show this last fact, you use that $\{D(f_i)\}$ is an open cover of an affine scheme if and only if $\sum(f_i)$ contains 1, where we note that an element of $\sum(f_i)$ is a FINITE sum, which gives us a finite sub-cover then. (Exercise, why is the first equivalence relation true).*

Definition 1.36. *We say a map of schemes $f : X \to Y$ is quasi-compact if $f^{-1}(V)$ is quasi-compact for all open affine sub-schemes V in Y.*

Proposition 1.37. *Let $f : X \to Y$ be a map of schemes. Assume $Y = \bigcup V_i$ is an open cover such that $f^{-1}(V_i)$ is quasi-compact and the V_i are affine, then f is quasi-compact. (This lets us check quasi-compactness on open affine covers).*

Proof. This proof has 2 steps

1. Take an open affine sub-scheme $V = \mathrm{Spec}(A) \subset Y$. Then $V \cap V_i \subset V$ is open which implies $V \cap V_i = \bigcup V_{ik}$, where $V_{ik} = D(f)$. This implies $V = \bigcup V_{ik}$. It suffices to show that $f^{-1}(V_{ik})$ is quasi-compact.

2. Let $f^{-1}(V_i)$ be the union of finite number of affine open sub-schemes $U_i j$. Then,

$$f^{-1}(V_i k) = \bigcup U_{ijk}$$

where $U_{ijk} = f^{-1}(V_i k) \cap U_i j$ and U_{ijk} is an open subset of U_{ij} of type $D(f)$. Then U_{ijk} is $\tau^{-1}(V_i k)$ where

$$\tau : U_{ij} \to V_i$$

is induced by f, U_{ij}, V_i are affine. Then, if $V_i = D(f)$, we get that $U_{ij} = D(\tau^*(f))$, where τ^* is the induced map of rings. Hence, U_{ijk} is affine, and $f^{-1}(V_{ik})$ is quasi-compact.

□

Now, we are ready to talk about morphisms of finite type.

Definition 1.38. *Let $f : X \to Y$ be a morphism of schemes. We say f is locally of finite type if for all affine open subsets $\mathrm{Spec}(A) = V \subset Y$ and for all affine open subsets $U \subset f^{-1}(V)$, the induced map*

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

induced by the restriction of f makes $\mathcal{O}_X(U)$ an $\mathcal{O}_Y(V)$-algebra of finite type. We say f is of finite type if it is both locally of finite type and quasi-compact.

Proposition 1.39. *Let $f : X \to Y$ be a morphism of schemes and $Y = \bigcup V_i$, where V_i are affine, such that for all i, $f^{-1}(V_i) = \bigcup U_{ij}$, where U_{ij} are affine,a nd $\mathcal{O}_X(U_{ij})$ is an $\mathcal{O}_Y(V_i)$-algebra of finite type. Then f is locally of finite type.*

Proof. We will see this later. □

Corollary 1.40. $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is of finite type if and only if the corresponding map $A \to B$ makes B into an A-algebra of finite type.*

Corollary 1.41. *Let $x \in X$. Then $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is not of finite type since $A_{\mathfrak{p}}$ is not an A-algebra of finite type in general. (Ex, $\mathbf{Z}$ and $\mathbf{Z}_{(p)}$*

And, at last, we are able to define what a variety is in the language of schemes:

Definition 1.42. *Let k be a field. A variety V over k is a k-scheme with structure morphism of finite type.*

2 Lecture on 4/2

The plan for this lecture is to cover the following:

1. Morphisms of finite type, finite morphisms
2. gluing and fibre products
3. noetherian schemes
4. reduced and integral schemes

2.1 Morphisms of Finite Type

2.1.1 Review and a Proposition

We recall a few definitions from last class.

Definition 2.1. *A morphism of schemes $f : X \to Y$ is quasi-compact if for every affine open subset $V = \mathrm{Spec}(A) \subset Y$, one has $f^{-1}(V)$ is quasi-compact.*

Remark 2.2. *We showed it is local over the base, i.e. it is enough to verify on an affine open cover of Y.*

Definition 2.3. *We say a morphism of schemes $f : X \to Y$ is locally of finite type if for every open affine subset $V = \mathrm{Spec}(A) \subset Y$, $U \subset f^{-1}(U)$ affine open, the induced map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ an $\mathcal{O}_Y(V)$-algebra of finite type.*

Definition 2.4. *We say a morphism of schemes $f : X \to Y$ is of finite type if it is locally of finite type and is quasi-compact.*

Proposition 2.5. *Locally of finite type is a property of morphisms of schemes which is local over the base: if $Y = \bigcup V_i$, where the V_i are affine open subsets and for all i, $f^{-1(V_i)} = \bigcup U_{ij}$, where U_{ij} are affine and $\mathcal{O}_X(U_{ij})$ is an $\mathcal{O}_Y(V_i)$-algebra of finite type, then f is locally of finite type.*

Before giving a proof of this, we give a few corollaries.

Corollary 2.6.

1. *$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type if B is an A-algebra of finite type.*
2. *The composition of morphisms of (locally) finite type is (locally) of finite type.*
3. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is locally of finite type, then f is locally of finite type.*

Now, in order to prove our proposition, we need a lemma:

Lemma 2.7. *Let A be a ring, and let B be an A-algebra.*

1. *If $\mathrm{Spec}(B) = \bigcup D(B_i)$ where B_{b_i} is an A-algebra of finite type, then B is an A-algebra of finite type.*

2. *If* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is an open immersion, then B is an A-algebra of finite type.*

Proof. 1) This proof is slightly more involved. First, we use that $\mathrm{Spec}(B)$ is quasi-compact to to get finitely many b_i such that $\mathrm{Spec}(B) = \bigcup_{finite} b_i$. !!!!!!!!!!!!!ISSUE Now, let B_{b_i} be an A-algebra generated by $\frac{a_{ij}}{b_i}$. Next, define C an A-algebra generated by b_i, a_{ij}, then $C \subset B$, C is of finite type as an A-algebra, and $C_{b_i} = B_{b_i}$. Since $\mathrm{Spec}(B) = \bigcup_{finite} b_i$, we have

$$1 = \sum_{finite} b_i b'_i, b'_i \in B$$

Now, let D be an A-algebra generated by C and the $b_i{}'$. Note that D is of finite type. We claim $B = D$. To see this, first we see that since for all k, $\mathrm{Spec}(B) = \bigcup D(b_i^k)$, $1 = \sum b_i^k d_{ik}$. But, for all $b \in B$, there exists k_i such that $b_i^k b = b_i^k c_i$ where $c_i \in C$. This implies

$$b = \sum_{i=1}^{m} (b_i^k c_i) d_{ik}$$

Hence, $b \in D$. This completes our proof.

2) To prove the second part of our lemma, we view $\mathrm{Spec}(B)$ as an open subset in $\mathrm{Spec}(A)$. Thus, we can cover $\mathrm{Spec}(B)$ as follows

$$\mathrm{Spec}(B) = \bigcup D(a_i) = \bigcup D(b_i)$$

where b_i is the image of a_i in B (notice we are covering $\mathrm{Spec}(B)$ in $\mathrm{Spec}(A)$ then using our map $A \to B$ to get a point in B). Then $B_{b_i} = A_{a_i}$ is of finite type over A. Hence, applying (1) we have our result. $\square$

Now, we can prove our proposition:

Proof. Let $V = \mathrm{Spec}(A) \subset Y, U = \mathrm{Spec}(B) \subset f^{-1}(A)$ be affine subsets. The goal is to show that B is an A-algebra of finite type.

We have that $Y = \bigcup V_i$, so $V \cap V_i = \bigcup V_{ik}$ where $V_{ik} = D(a_{ik})$ are open affine subsets. By hypothesis, $f^{-1}(V_i) = \bigcup U_{ij}$ which are affine. We have $\mathcal{O}_X(u_{ij}) = B_{ij}$ is an $\mathcal{O}_Y(V_i)$-algebra of finite type. Let $u_{ijk} = f^{-1}(V_{ik}) \cap u_{ij}$. Then we have that $\mathcal{O}_X(u_{ijk}) = (B_{ij})_{b_{ijk}}$ is a principal open set in u_{ij}. We see V_{ij} corresponds to $D(a_{ik})$, so $A_{a_{ik}}$. As B_{ij} is an A_i-algebra of finite type, one has $(B_{ij})_{b_{ijk}}$ is an $(A_i)_{a_{ik}}$-algebra of finite type (where $a_{ik} \mapsto b_{ijk}$).

Now, by construction, $f^{-1}(V) = \bigcup U_{ijk}$. Also, $\mathcal{O}_X(u_{ijk})$ is of finite type over $\mathcal{O}_Y(V_{ik})$, so also over $\mathcal{O}_Y(V)$ (by our above lemma part (b)).

At last, let $U = \mathrm{Spec}(B) \subset f^{-1}(A)$ be as in the beginning. For all $x \in U$, choose a neighborhood U_x of type $D(b_x)$ with $B_{b_i} \subset U_{ijk}$ for some i, j, k. By our lemma, B_{b_i} is of finite type over $\mathcal{O}_X(U_{ijk})$ hence also over $\mathcal{O}_Y(V)$ by our result above. As U is affine, it is also quasi-compact. We have U covered by finitely many U_x, so by our lemma part (a) we are done! $\square$

2.1.2 Affine Morphisms, finite morphisms

Definition 2.8. *A morphism of schemes $f : X \to Y$ is affine if for all $V \subset Y$ affine open subsets, then $f^{-1}(V)$ is an affine open subset.*

Definition 2.9. *Let $f : X \to Y$ be a morphism of schemes. We say it is if it is finite and if for all $V \subset Y$ affine open subsets, $\mathcal{O}_X(f^{-1}(V))$ is a finite $\mathcal{O}_Y(V)$-algebra (i.e. it is finitely generated as a module which is more restrictive).*

Example 2.10. $\frac{k[x]}{(p)}$ *is a k-module of finite type. (notice our inclusion map $k \to \frac{k[x]}{(p)}$ which gives us a map of schemes as desired).*

Proposition 2.11.

- *finite/affine morphisms are stable by composition.*
- *Let $f : X \to Y$ be a morphism of schemes, $Y = \bigcup V_i$ an affine cover such that $U_i = f^{-1}(V_i)$ are affine, then f is affine. If also, $\mathcal{O}_X(U_i)$ is an $\mathcal{O}_Y(V_i)$-module of finite type, then f is finite.*

2.2 Gluing, Fibred Products

First, we want to figure out how to glue together some schemes we may have in a similar way to how we might glue together manifolds.

2.2.1 Main Properties

Lemma 2.12. *Let S be a scheme and $\{X_i\}$ a family of S-schemes. Let $\{X_{ij}\}$ be a family of open sub-schemes of X_i and $f_{ij} : X_{ij} \to X_{ji}$ be S-morphisms such that*

1. $f_{ii} = Id$
2. $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$
3. $f_{ik} = f_{jk} \circ f_{ij}$ *on* $X_{ij} \cap X_{ik}$

Then there exists an S-scheme X (unique up to isomorphism) and open immersions $g_i : X_i \to X$ such that $g_i = g_j \circ f_{ij}$ on X_{ij} with $X = \bigcup g_i(X_i)$. We call X the gluing of X_i along X_{ij}

Definition 2.13. *Let X, Y be S-schemes. Let $X(Y) = \mathrm{Hom}_S(Y, X)$. (These are the Y-rational points of X)*

For our product of schemes, we want $(X \times Y)(W) = X(W) \times Y(W)$. Hence, we get the following definition:

Definition 2.14. *Let X, Y be S-schemes. The fibre product $Z = X \times_S Y$ is an S-scheme satisfying the following universal property: We have projection maps $p_X : Z \to X$ and $p_Y : Z \to Y$ such that for any S-morphisms f, g where $f : W \to X$*

and $g : W \to Y$, there exists a unique S-morphism $W \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
W \arrow[dr, "{(f,g)}"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Z = X \times_S Y \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \\
& X &
\end{tikzcd}$$

Theorem 2.15. *$X \times_S Y$ exists and is unique up to unique isomorphism.*

Proof. Sketch First, we consider the affine case. Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(C)$. We claim $X \times_S Y = \mathrm{Spec}(A \otimes_C B)$ where the projection maps are induced by the canonical homomorphisms $A \to A \otimes_C B$ and $B \to A \otimes_C B$. From the universal property of the tensor product and the correspondence of rings and affine schemes, the following diagram solves our problem:

$$\begin{tikzcd}
& D & \\
& A \otimes_C B \arrow[u] & \\
A \arrow[ur] \arrow[uur, bend left] & & B \arrow[ul] \arrow[uul, bend right]
\end{tikzcd}$$

Now, the general case follows from the affine case by gluing morphisms. $\square$

Now, we give some properties of the fibre product:

Proposition 2.16.
- $X \times_S S \cong X$
- $X \times_S Y \cong Y \times_S X$
- $(X \times_S Y) \times_S Z \cong X \times_S (Y \times_S Z)$
- $(X \times_S Y) \times_Y Z \cong X \times_S Z$
- *If $U \subset X$ is open, then $U \times_S Y$ is the inverse image of U via the projection $p : X \times_S Y \to X$.*
- *If $U \subset S$ open such that $f(X) \subset U$, then $X \times_S Y \cong X \times_U Y_U$, where $Y_U = g^{-1}(U)$ and g is the structure morphism of Y, which we know define...*

Definition 2.17. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. The fibre of f in Y is the $\kappa(y)$-scheme $X_y = X \times_Y \operatorname{Spec}(\kappa(y))$. If $y \in Y$ is the generic point, then X_y is called the generic fibre.*

Lemma 2.18. *Using the same notation as above, the map $p : X_y \to X$ induces a homeomorphism of X_y onto $f^{-1}(y)$.*

Proof. From our last property above, we can change Y to an open affine subset $V = \operatorname{Spec}(A) \ni y$. Similarly, if $U \subset X$ is open, then $p^{-1}(U) = U \times_Y \operatorname{Spec}(\kappa(y))$. Hence, we can assume $X = \operatorname{Spec}(B)$. Now, let $\mathfrak{p} in \operatorname{Spec}(A)$ correspond to y. Then $\mathfrak{p}$ is induced by

$$\phi : B \to B \times_A \kappa(A)$$

which is really just

$$\phi : B \to B \otimes_A A_{\mathfrak{p}} \to B \otimes_A \kappa(\mathfrak{p}) \cong B/\kappa(\mathfrak{p})B$$

This is just equivalent to the prime ideals $I \subset B$ containing $\kappa(\mathfrak{p})B$ which is equivalent $\kappa^{-1}(I) = \mathfrak{p}$. □

2.2.2 Base Change

Definition 2.19. *Let X be an S-scheme. For T another S-scheme, we say $X \times_S T$ is obtained by base change which is a scheme over T.*

Definition 2.20. *We say a property of morphisms is stable under base change if whenever $X \to Y$ has the property then $X \times_Y Y'$ has the property too for ANY $Y' \to Y$.*

Example 2.21. *Here are a few properties stable under base change:*

- *closed, open, and locally closed immersions*
- *quasi-compact*
- *universally closed*
- *(quasi-)separated*
- *monomorphism, surjective, universally injective*
- *affine, quasi-affine, (locally) of finite type*

Remark 2.22. *Many of these are also stable under composition.*

2.3 Noetherian Schemes

Definition 2.23. *A topological space X is noetherian if every decreasing sequences of closed subsets is eventually constant (stabilizes).*

Proposition 2.24. *Let A be a noetherian ring. Then $X = \operatorname{Spec}(A)$ is a Noetherian topological space.*

Proof. Let A be a noetherian ring. We know $V(I) \subset V(J)$ if and only if $J \subset \sqrt{I}$. Thus, any descending chain of closed subsets corresponds to an ascending chain of ideals in A, which must terminate! $\square$

Warning 2.25. *The converse is false in general. There exists local non-noetherian rings. Consider the ring*

$$\frac{k[x_1, \ldots]}{(x_1^2, \ldots)}$$

The spectrum of this ring has exactly one point, the maximal ideal $(x_1, \ldots)$. But clearly the ring is not noetherian.

Fact 2. *In general,* $\operatorname{Spec}(A)$ *is a noetherian topological space if and only if A satisfies the ascending chain condition for* <u>*radical ideals*</u>*.*

Fact 3. *Topology Facts*

1. *An open subset of a noetherian topological space is noetherian*
2. *If $X = \bigcup_{finite} X_i$ is an open cover where X_i is noetherian for all i, then X is noetherian*
3. *A topological space X is noetherian if and only if every open subset of X is quasi-compact*

Definition 2.26. *A topological space X is irreducible if it is non-empty and if $X = X_1 \bigcup X_2$, where X_1, X_2 are closed, implies $X = X_1$ or $X = X_2$.*

Fact 4. *More Topology Facts*

1. *An open subset of an irreducible space X is irreducible*
2. *If X is a noetherian topological space, them there exists a unique up to permutation decomposition $X = \bigcup Y_i$, where Y_i are irreducible closed subsets that are not properly contained in each other.*

Definition 2.27. *A scheme is connected if its topological space is connected. A scheme is irreducible if its topological space is irreducible (connected just replaces closed subsets with open ones in the definition of irreducible).*

Definition 2.28. *A scheme X is locally noetherian if there exists an open cover $X = \bigcup \operatorname{Spec}(A_i)$ where every A_i is noetherian. We say X is noetherian if it is also quasi-compact.*

Fact 5. $\operatorname{Spec}(A)$ *is a noetherian scheme if and only if A is noetherian.*

2.4 Reduced and Integral Schemes

Definition 2.29. *Let X be a scheme. We say X is reduced if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced, i.e. has no nonzero nilpotents.*

Example 2.30. $\operatorname{Spec}(\frac{k[t]}{t^2})$ *is not reduced (can see this easily via the next proposition and looking at the global sections).*

Proposition 2.31. *A scheme X is reduced if and only if for all open subsets $U \subset X$, $\mathcal{O}_X(U)$ is reduced.*

Proof. ($\Longleftarrow$) Take $x \in X$ which corresponds to an element $\mathfrak{p} \in \operatorname{Spec}(A)$ ($\mathcal{O}_{X,x} = A_{\mathfrak{p}}$). If $\frac{f}{g} \in A_{\mathfrak{p}}$ is nilpotent, then there exists $h \notin \mathfrak{p}$, $m \geq$ such that $hf^m = 0$. This is equivalent $(hf)^m = 0$ which is equivalent to $hf = 0$ since A is reduced. Hence, $\frac{f}{g} = 0$.

($\Longrightarrow$) Now, in the opposite direction, if $f \in \mathcal{O}_X(U)$ is nilpotent for some open subset $U \subset X$, then $f_x \in \mathcal{O}_{X,x}$ is zero for all $x \in \mathcal{O}_{X,x}$. This implies $f = 0$ since $\mathcal{O}_X$ is a sheaf. $\square$

Remark 2.32. *Let A be a ring. Let $A_{red} = A/nilA$, where $nilA$ is just the nilpotent elements of A. Then* $\operatorname{Spec}(A_{red}) = \operatorname{Spec}(A)$ *as topological spaces (since nilpotents are contained in every prime ideal). We call* $\operatorname{Spec}(A_{red})$ *the reduced structure.*

Definition 2.33. *A scheme X is integral if X is irreducible and reduced.*

Proposition 2.34. *An affine scheme* $\operatorname{Spec}(A)$ *is integral if and only if A is an integral ring.*

Definition 2.35. *Let k be a field, and X be a scheme over k. We say X/k is geometrically integral if $X \times_k \bar{k}$ is integral. (Geometrically "blank" if the property is still true after doing a base change).*

2.5 Separated Morphisms

A problem we run into is that $\operatorname{Spec}(A)$ is almost never Hausdorff as open sets are dense. We need a better notion then...

Remark 2.36. *The Zariski topology on* $\operatorname{Spec}(A)$ *is Hausdorff if and only if A is zero-dimensional (which means every prime ideal is a maximum ideal and thus closed)*

Definition 2.37. *Let $f : X \to Y$ be a morphism of schemes. We define*

$$\Delta_{X/Y} : X \to X \times_Y X$$

which is induced by (Id_x, Id_x). We call it the diagonal morphism.

Definition 2.38. *Using the same notation as above, we say f is separated if Δ is a closed immersion. We say an S-scheme X is separated if its structure morphism is separated.*

Proposition 2.39. *Any morphism between affine schemes is separated.*

Proof. Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Suppose we have a morphism $X \to Y$ which is the same as a map of rings $A \to B$. Then we see that

$$\Delta : X \to X \times_Y Y$$

is the same as

$$B \otimes_A B \to B$$

given by

$$b \otimes b' \mapsto bb'$$

which is clearly surjective (just take 1 as one input). □

Fact 6. *Let* $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a closed immersion if and only if the induced map* $f : A \to B$ *is surjective*

Proof. First, suppose $g : A \to B$ is surjective. We want to show the induced map of affine schemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a homeomorphism from $\mathrm{Spec}(B)$ to a closed subset of $\mathrm{Spec}(A)$ and $f^{\#} : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ is surjective. Now, this last part is equivalent to showing the map is surjective on stalks. Recall our induced map of stalks in the affine case is

$$A_{f(\mathfrak{p})} \mapsto B_{\mathfrak{p}}$$

which is induced by g. This is in fact surjective.

Now we prove the forward direction. Suppose $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed immersion. $\square$

3 Lecture on 4/9

The plan for this lecture is

- Separated Morphisms
- Dimension (at the end)
- Proper morphisms, projective morphisms

Note: We only covered up to proper morphisms because of a fire alarm.

3.1 Separated Morphisms

Remark 3.1. *We will give some motivation for the definition of separated morphisms of schemes. Let $f : X \to Y$ be a continuous map of topological spaces. We define*

$$\Delta_{X/Y} : X \to X \times_Y X$$

to be the unique map such that its composition with the usual projection maps is the identity. We say that the map f is separated if the image of the diagonal morphism is closed. Moreover, we see that a map f of topological spaces is separated if and only if any two distinct points which are identified by f can be separated by disjoint open sets in X. In particular, a space X is Hausdorff if and only if the unique map $X \to 1$ is separated.

With this in mind, we recall the definition of a separated morphism:

Definition 3.2. *Let $f : X \to Y$ be a morphism of schemes. We define*

$$\Delta_{X/Y} : X \to X \times_Y X$$

which is induced by (Id_x, Id_x). We call it the diagonal morphism.

Definition 3.3. *Using the same notation as above, we say f is separated if Δ is a closed immersion. We say an S-scheme X is separated if its structure morphism is separated.*

Now, we explore an example of a non-separated morphism:

Example 3.4. *We let*

- *k be a field*
- $X_1 = X_2 = \mathbb{A}^1_k$
- *$U_1 = U_2 = \mathbb{A}^1_k - \{\mathfrak{p}\}$, where $\mathfrak{p} = (x)$ i.e. the affine line minus the origin*
- *X be the scheme obtained by gluing X_1 and X_2 along the morphism $U_1 \cong U_2$ (the identity map)*

Then, X is NOT separated. To see why, look at U the image of X_1 in X and V the image of X_2 in X. We see that $\mathcal{O}_X(U \cap V) = k[t, \frac{1}{t}]$, but $\mathcal{O}_X(U) = \mathcal{O}_X(V) = k[t]$. Thus, the intersection $U \cap V$ is NOT affine. By a proposition we will prove soon, this implies X cannot be separated.

We have already defined what a variety is in general in lecture 1. Now, we define what an algebraic variety is

Definition 3.5. *An algebraic variety is a separated scheme of finite type over a field k.*

Proposition 3.6. *Being separated is local over the base: if $f : X \to Y$ is a homomorphism, $Y = \bigcup Y_i$ an open cover, and the maps $f^{-1}(Y_i) \to Y_i$ are separated, then f is separated.*

Remark 3.7. *In particular, this means that all morphisms of affine schemes are separated.*

Proof. We will only sketch the proof. The idea is that it is enough to check that the image of Δ is closed, and we do this locally on the Y_i. □

Proposition 3.8. *Let $S = \operatorname{Spec}(C)$ and X be an S-scheme. Then the following are equivalent:*

1. *X is separated over S*
2. *If $U, V \subset X$ are affine open subsets, then $U \cap V$ is affine and*

$$\varphi_{UV} : \mathcal{O}_X(U) \otimes_C \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \text{ by } \\ s \otimes t \mapsto s|_{U\cap V} \times t|_{U \cap V}$$

   is surjective.

3. *There exists an open cover $X = \bigcup U_i$, where the U_i are affine, such that for all i, j, $U_i \cap U_j$ is affine and the map*

$$\mathcal{O}_X(U_i) \otimes_C \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

   is surjective.

Proof. Again, we will only sketch the proof of this. The main idea is that $\Delta^{-1}(U \times_S V) = U \cap V$, so the condition corresponds to

$$f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

is surjective for f closed immersions. □

Example 3.9. *We have that $\operatorname{Proj}(k[T_0, \dots, T_n])$ is separated over k, a field. This is because $D_+(T_i) \cap D_+(T_j) = D_+(T_iT_j)$ and this is surjective, so we can apply our proposition from above.*

Proposition 3.10.

- *Open and closed immersions are separated.*
- *The composition of two separated morphisms is separated.*
- *Separated morphisms are stable under base change.*
- *If $g \circ f$ is separated, then f is separated. In particular, any k-morphism of algebraic varieties is separated.*

3.1.1 Important Property

Theorem 3.11. *Let S be a scheme, X be an S-scheme, and Y be a separated S-scheme. Let $f, g : X \to Y$ be two S-morphisms. If X and Y coincide on an open dense subset $U \subset X$, then $f = g$.*

Proof. Let

$$\Delta : Y \to Y \times_S Y$$

be the diagonal of the structure morphism of Y. Also, let

$$h = (f, g) : X \to Y \times_S Y.$$

Then, we have that $\Delta \circ f = (f, f)$, so $h = \Delta \circ f$ on U. This means $U \subset h^{-1}(\Delta(Y))$. But $h^{-1}(\Delta(Y))$ is closed, so Y is separated which means $\Delta(Y)$ is closed. Lastly, $X \subset h^{-1}(\Delta(Y))$, so $f = g$ as maps of sets.

Now, we need to show they agree as maps of schemes. We may assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. $\varphi, \psi : A \to B$ correspond to f, g. Let $a \in A, b = \varphi(a) - \psi(a)$. As the restrictions of b to $\mathcal{O}_X(U)$ is zero, we have $U \subset V(bB)$. This implies $V(bB) = \mathrm{Spec}(B)$ since $V(bB)$ is closed and dense. Hence, b is nilpotent and thus 0 since X is reduced. Thus, $\varphi = \psi$. $\square$

3.2 Proper Morphisms

Proper morphisms will provide an analogue of compactness in the setting of schemes.

Remark 3.12. *In more detail, proper maps really extend the idea of proper maps of complex analytic spaces. We have an analogy between proper as "universally closed map from Hausdorff to locally compact Hausdorff" and proper as "universally closed map of finite type and separated". Both definitions are universally closed + finiteness assumption + separation assumption.*

Definition 3.13. *Let $f : X \to Y$ be a morphism of schemes. We say it is closed if f sends closed subsets to closed subsets. We say f is universally closed if for all morphisms $Y' \to Y$, after base change, $X \times_Y Y' \to Y'$ is closed.*

Definition 3.14. *A morphism of schemes $f : X \to Y$ is proper if f is of finite type, separated, and universally closed.*

Example 3.15.

- *Closed immersions are proper.*
- *$\mathbb{A}^1_k$ is NOT proper since $\mathbb{A}^2_k = \mathbb{A}^1_k \times A^1_k \to \mathbb{A}^1_k$ is NOT closed as the image of the closed subset $xy = 1$ is $\mathbb{A}^1_k - \{0\}$.*

Proposition 3.16.

- *Finite morphisms are proper.*

- *The composition of proper morphisms is proper.*
- *Proper morphisms are stable under base change.*
- *If $g \circ f$ is proper and g is separated, then f is proper. In particular, any k-morphism between proper algebraic k-varieties is proper.*

4 Lecture on 4/16

The plan for this lecture is

- Valuative criterion of properness and applications to projective morphisms
- Dimension of schemes
- Normal schemes

4.1 Valuative Criterion for Properness

Definition 4.1. *Let R be an integral domain and K its field of fractions. We say R is a valuation ring if for all $x \in K - \{0\}$, one has $x \in R$ or $x^{-1} \in R$.*

Fact 7. *We have the following results from commutative algebra:*

1. *If R is a valuation ring, then there exists $(\Gamma, +)$ an abelian totally ordered group and a map $\nu : K - \{0\} \to \Gamma$ such that*
   - $\nu(x) \geq 0$ *if and only if* $x \in R$
   - $\nu(xy) \geq \nu(x) + \nu(y)$
   - $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$
2. *R is noetherian if and only if one can take $\Gamma = \mathbb{Z}$.*

With this, we can state our valuative criterion for properness:

Theorem 4.2. *Valuative Criterion for Properness Let $f : X \to Y$ be a morphism of finite type. Assume X is noetherian. Then f is proper if and only if the following condition is satisfied: let R be a valuation ring and K be its field of fractions. For every commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "g"] \arrow[d, "I"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a unique extension $\mathrm{Spec}(R) \to X$ of g such that the whole diagram still commutes (the dotted line is this extension).

Proof. See Hartshorne chapter 2 section 5. $\square$

An example of an application of this seemingly odd theorem deals with projective morphisms...

Definition 4.3. *Let Y be a scheme. We define the projective n-space over Y to be*

$$\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$$

Remark 4.4. *We recall that* $\mathrm{Spec}(\mathbb{Z})$ *is a terminal object in the category of schemes, so this definition is well defined (i.e. the fibre product actually exists).*

Definition 4.5. *We say that a morphism* $f : X \to Y$ *of noetherian schemes is projective if* f *factors as*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & \mathbb{P}^n_Y \arrow[r, "p"] & Y \end{tikzcd}$$

where i *is a closed immersion, and* p *is projection on* Y.

Example 4.6. $\mathrm{Proj}(\frac{A[x_0,\ldots,x_n]}{I})$

Proposition 4.7. *A projective morphism of notherian schemes is proper.*

Proof. LONG!!!!!!!!!!!!!!!!!! ☐

4.2 First Notions of Dimension

We recall a few ideas from commutative algebra:

1. For a prime ideal $\mathfrak{p} \subset A$, we define the height of $\mathfrak{p}$ to be

$$\mathrm{ht}(\mathfrak{p}) = \max\{n : \exists \ \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p} \ \text{ with } \mathfrak{p}_i \text{ prime}\}$$

2. The dimension of a ring A is

$$\dim A := \sup_{\mathfrak{p} \subset A \ \text{ prime}} (\mathrm{ht}(\mathfrak{p}))$$

   For example, a field has dimension zero since it has no nontrivial ideals.

3. If A is noetherian, then $\dim A[x_1, \ldots, x_n] = \dim A + n$
4. If A is an algebra of finite type over a field k and an integral domain, then

$$\dim A = \mathrm{tr.\,deg.}_k \mathrm{Frac}(A)$$

   Also, for all $\mathfrak{p} \subset A$, one has

$$\dim A = \mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p})$$

Now, we define the dimension of a scheme. It turns out the proper definition is just the dimension of the underlying topological space.

Definition 4.8. *Let* X *be a nonempty topological space. Then we define*

$$\dim X := \sup\{n : \exists \ \textit{a sequence of irreducible closed subsets } \ Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n\}$$

Proposition 4.9. *If* $X = \mathrm{Spec} A$*, then* $\dim X = \dim A$.

Proof. An irreducible closed subset of X corresponds to a $V(\mathfrak{p})$, where $\mathfrak{p}$ is prime. Thus, a chain of prime ideals corresponds to a chain of irreducible closed subsets, so we are done! ☐

Proposition 4.10. *Let X be a topological space. Then*

1. *For all subsets* $Y \subset X$*, we have* $\dim Y \leq \dim X$.
2. *If* X *is irreducible of finite dimension, and if* $Y \subset X$ *is a closed subset, then if* $\dim X = \dim Y$*, we have* $X = Y$.
3. *If* X *is noetherian, then* $\dim X =$ *maximum of dimensions of irreducible components.*
4. *If* $\{U_i\}$ *is an open cover of* X*, then* $\dim X = \sup \dim U_i$.

Proof. !! $\square$

Example 4.11.

1. *An irreducible scheme of dimension zero is a point (i.e.* $\mathrm{Spec} k$*).*
2. $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$*. !!!!!!!!!!!!!!!!!*

Definition 4.12. *Let* Y *be a topological space inside of a topological space* X*. Assume* Y *is closed. Then* $\mathrm{codim}(Y, X)$ *of* Y *in* X *is the supremum of the lengths of chains* $Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$*, where the* Y_i *are closed irreducible subsets.*

Example 4.13. *In* $\mathrm{Spec} A$*, the codimension of* $V(\mathfrak{p})$*is just the height of the the prime ideal (just reverse inclusions when switching between geometry and algebra).*

Now, for a scheme of finite type of a field, we have

Theorem 4.14. *Let* X *be an integral scheme of finite type over a field* k*. Then we have*

1. $\dim X = \dim$ *???????????????????*
2.
3.
4.

Fact 8. *For morphisms,*

1. *if* $X \to Y$ *a finite surjective morphism of schemes, then* $\dim X = \dim Y$.
2. *if* $f : X \to Y$ *is flat, then* $\dim X_y = \dim X - \dim Y$*. (flat just means the local ring* $\mathcal{O}_{X,x}$ *is a flat module over* $\mathcal{O}_{Y,f(x)}$*)*

4.3 First Notions of Regularity: Normal Schemes

Definition 4.15. *We say that a ring* A *is normal if it is an integral domain and if* A *is integrally closed in its field of fractions.*

Example 4.16. *From a homework in Algebra II, we know a UFD is integrally closed, so it is normal.*

Definition 4.17. *We say that a scheme* X *is normal at* $x \in X$ *if the local ring* $\mathcal{O}_{X,x}$ *is normal. We say* X *is normal if it is normal at every point.*

Fact 9. $X = \mathrm{Spec} A$ *is normal if and only if* A *is a normal ring. ?????????????????????*